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A NOTE ON DISSIPATION IN FREE-CONVECTION

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NOMENCLATURE

- c_p, specific heat at constant pressure;
- x, y, distances along and perpendicular to the plate;
- u, v, velocity components along x and y directions;
- Gr_x, local Grashof number defined by (2);
- g, acceleration due to gravity;
- k, thermal conductivity;
- t, temperature;
- f, non-dimensional stream-function;
- F, transformed stream-function in the inner layer;
- G, transformed stream-function in the outer layer;
- T, function of x defined by (2);
- q'', rate of heat-flux at the surface.

Greek symbols

- γ, a constant;
- ψ, stream-function;
- σ, Prandtl number;
- ϵ, local dissipation number defined by (2);
- β, co-efficient of volume expansion of the fluid;
- η, similarity variable defined by (2);
- ν, kinematic viscosity;
- α, thermal diffusivity;
- θ, temperature excess;
- ϕ, non-dimensional temperature function defined by (2);
- ζ, stretched similarity variable;
- Φ, transformed temperature function.

Subscripts

- w, wall condition;
- ∞, condition at large distance from the plate;
- 0, no dissipation;
- 1, first order dissipation effects.

INTRODUCTION

GEBHART [1] investigated effects of viscous dissipation in natural convection about semi-infinite flat vertical surfaces subject to both isothermal and uniform heat-flux surface conditions. He used a perturbation method and calculated the first temperature perturbation function for the Prandtl numbers (σ) 10^{-2}, 0·72, 10^2 and 10^4 for the former case and for 10^2 for the latter. For the isothermal case, he put forward a conjecture regarding the asymptotic behaviour of the first temperature perturbation function as $\sigma \to \infty$. Roy [2], using the double-boundary-layer concept first introduced by Stewartson and Jones [3], obtained solutions for isothermal surfaces in powers of σ, for large σ, to terms $0(\sigma^{-\frac{1}{2}})$, and substantiated the above-mentioned conjecture.

In this paper, we propose to solve the high Prandtl number problem for uniform heat-flux surface conditions using a similar technique. The solutions will be obtained to terms $0(\sigma^{-1})$. It will be seen that there still exists an asymptotic behaviour of the first temperature perturbation function.

ANALYSIS

The well-known equations for conservation of mass, momentum and energy are

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \pm g\beta\theta, \\ &u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \end{aligned}\right\} \quad (1)$$

subject to the boundary conditions

$$u = v = 0, \qquad q'' = -k\frac{\partial \theta}{\partial y}, \qquad y = 0,$$

$$u \to 0, \qquad \theta \to 0 \qquad \text{as} \quad y \to \infty.$$

In the above, u, v are the velocity components, θ is the temperature excess $(t - t_\infty)$, x is measured from the leading edge along the plate and y is the distance out perpendicular to the plate, q'' is the rate of heat-flux at the surface, the plus and minus signs apply for heating and for cooling of the fluid respectively and the other symbols have their usual meanings.

Perturbation-type similar solutions of (1) are given by

$$\left.\begin{aligned} u &= \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}, \\ \phi &= T(t_\infty - t) = \phi_0 \pm 5\epsilon\phi_1 \pm \ldots, \\ \psi &= \nu 5^{4/5}(Gr_x)^{1/5}(f_0 \pm 5\epsilon f_1 + \ldots), \\ Gr_x &= \left|\frac{g\beta x^4 q''}{k\nu^2}\right|, \\ \eta &= \frac{y}{x}\left(\frac{Gr_x}{5}\right)^{1/5}, \\ \epsilon &= \frac{g\beta x}{c_p}, \\ T(x) &= \frac{k}{xq''}\left(\frac{Gr_x}{5}\right)^{1/5}, \end{aligned}\right\} \tag{2}$$

where f's and ϕ's are functions of η alone. We further assume, with Gebhart, that $f_1 = 0$. Thus the equations to be satisfied by f_0, ϕ_0 and ϕ_1 are

$$f_0''' + 4f_0 f_0'' - 3(f_0')^2 - \phi_0 = 0, \tag{3}$$

$$\phi_0'' + \sigma(4f_0\phi_0' - f_0'\phi_0) = 0, \tag{4}$$

$$\phi_1'' + \sigma(4f_0\phi_1' - 6f_0'\phi_1 - f_0''^2) = 0, \tag{5}$$

together with the boundary conditions

$$\left.\begin{aligned} &f_0(0) = f_0'(0) = f_0'(\infty) = \phi_0'(0) - 1 = \\ &\phi_0(\infty) = \phi_1'(0) = \phi_1(\infty) = 0. \end{aligned}\right\} \tag{6}$$

It should be noted that what appears as ϕ in Gebhart [1] is $-\phi$ in our notations, in line with Sparrow and Gregg [4] who solved the no-dissipation problem for discrete values of σ.

INNER AND OUTER LAYERS

For large values of σ, the effects of temperature variation are confined to a very thin layer lying well within the velocity boundary layer. In the case of isothermal surfaces, their thicknesses are $0\,(\sigma^{-1/4})$ and $0\,(\sigma^{1/4})$ respectively [2, 3]. However, in the present case these are $0\,(\sigma^{-1/5})$ and $0\,(\sigma^{\frac{3}{10}})$, their ratio remaining $\sigma^{-1/2}$ as in the former case. The appropriate transformations are:

Inner layer

$$\left.\begin{aligned} \zeta_1 &= \sigma^{1/5}\eta, \\ f_0 &= \sigma^{-4/5}F_0(\zeta_1), \\ \phi_0 &= \sigma^{-1/5}\Phi_0(\zeta_1), \\ \phi_1 &= \sigma^{-1/5}\Phi_1(\zeta_1); \end{aligned}\right\} \tag{7}$$

Outer layer

$$\left.\begin{aligned} \zeta_2 &= \gamma\sigma^{-\frac{3}{10}}\eta, \\ f_0 &= \gamma\sigma^{-\frac{3}{10}}G_0(\zeta_2), \\ \phi_0 &= \phi_1 \doteq 0; \end{aligned}\right\} \tag{8}$$

where γ is a suitable constant to be specified later. The corresponding equations for F_0, Φ_0, Φ_1 and G_0 are

$$F_0''' - \Phi_0 + \sigma^{-1}\{4F_0F_0'' - 3(F_0')^2\} = 0, \tag{9}$$

$$\Phi_0'' + 4F_0\Phi_0' - F_0'\Phi_0 = 0, \tag{10}$$

$$\Phi_1'' + 4F_0\Phi_1' - 6F_0'\Phi_1 - (F_0'')^2 = 0, \tag{11}$$

$$G_0''' + 4G_0G_0'' - 3(G_0')^2 = 0. \tag{12}$$

In the above, a prime denotes differentiation with respect to the appropriate variable, ζ_1 or ζ_2. The boundary conditions at $\eta = \infty$ are redundant for F_0, and those at $\eta = 0$ for G_0. Further, equations (9)–(12) suggest that series solutions in some negative powers of σ exist. The appropriate series and the boundary conditions at $\zeta_1 = \infty$ and at $\zeta_2 = 0$ are determined by matching the inner solutions for large values of ζ_1 with the outer solutions for small values of ζ_2. They are

$$\left.\begin{aligned} F_0 &= \sum_{i=0}^{\infty} \sigma^{-i/2} F_{0i}, \\ \Phi_0 &= \sum_{i=0}^{\infty} \sigma^{-i/2} \Phi_{0i}, \\ \Phi_1 &= \sum_{i=0}^{\infty} \sigma^{-i/2} \Phi_{1i}, \\ G_0 &= \sum_{i=0}^{\infty} \sigma^{-i/2} G_{0i}; \end{aligned}\right\} \tag{13}$$

and

$$F_{00}''(\infty) = 0,$$

$$F_{01}''(\infty) = \gamma^3 G_{00}''(0),$$

$$F_{02}''(\infty) = \lim \zeta_1 \to \infty\, \gamma^3\{G_{01}''(0) + 3\gamma\zeta_1\},$$

$$G_{00}(0) = 0,$$

$$G_{00}'(0) = 1, \qquad \text{choosing} \quad \gamma^2 = \lim \zeta_1 \to \infty\, F_{00}'(\zeta_1)$$

$$G_{01}(0) = \lim \zeta_1 \to \infty\, \{1/\gamma F_{00}(\zeta_1) - \gamma\zeta_1\},$$

$$G_{01}'(0) = \lim \zeta_1 \to \infty\, \{1/\gamma^2 F_{01}'(\zeta_1) - \gamma\zeta_1 G_{00}''(0)\},$$

$$G_{02}(0) = \lim \zeta_1 \to \infty\, \{1/\gamma F_{01}(\zeta_1) - \gamma\zeta_1 G_{01}'(0) - \gamma^2/2\zeta_1^2 G_{00}''(0)\},$$

$$G_{02}'(0) = \lim \zeta_1 \to \infty\, \{1/\gamma^2 F_{02}'(\zeta_1) - \gamma\zeta_1 G_{01}''(0) - 3/2\gamma^2\zeta_1^2\}.$$

SOLUTIONS

The twelve equations that are obtained from (9)–(13) have been solved on the computer. The unknown boundary conditions required to start numerical integrations are given below:

$$F_{00}''(0) = 0{\cdot}811546, \qquad \Phi_{00}(0) = -1{\cdot}147565,$$

$F''_{01}(0) = -0.173879, \quad \Phi_{01}(0) = -0.226844,$

$F''_{02}(0) = 0.134655, \quad \Phi_{02}(0) = 0.030392,$

$G_{00}(0) = 0.0, \quad G'_{00}(0) = 1, \quad G''_{00}(0) = -1.837319,$

$G_{01}(0) = -0.300701, \quad G'_{01}(0) = 0.732916,$

$G''_{01}(0) = -1.399446, \quad G_{02}(0) = -0.311263,$

$G'_{02}(0) = 0.804204, \quad G''_{02}(0) = -1.578787,$

$\Phi_{10}(0) = -0.118237, \quad \Phi_{11}(0) = 0.077670,$

$\Phi_{12}(0) = -0.094576.$

CONCLUSIONS

The effect of viscous dissipation in the case of an assigned surface heat-flux is to make necessary a larger difference between t_w and t_∞ for the covection of a heat-flux to the fluid. The surface temperature is given by the relationship

$$t_w - t_\infty = -\frac{q''x/k}{(\sigma Gr_x/5)^{\frac{1}{5}}} \times [1.147565 + 0.226844\sigma^{-\frac{1}{2}} - 0.030392\sigma^{-1}][1 + 5r\epsilon],$$

where

$$r = \frac{\phi_1(0)}{\phi_0(0)} = 0.103033 - 0.088049\sigma^{-\frac{1}{2}} + 0.102548\sigma^{-1}.$$

The important ratio r has the values 0.085444, 0.095253, 0.100351 and 0.102163 for $\sigma = 10, 10^2, 10^3$ and 10^4 respectively. The value given by Gebhart [1] for $\sigma = 10^2$, namely 0.09547, compares very well with the corresponding value obtained above. However, though r increases with σ it has an asymptotic value 0.103033 for $\sigma = \infty$.

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NOTE ADDED IN PROOF

A more recent paper is B. Gebhart and J. Mollendorf, Viscous dissipation in external natural convection flows, *J. Fluid Mech.* **38**, 97–107 (1969).

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PREDICTION OF FLOW AND HEAT TRANSFER IN TURBULENT CYLINDRICAL WALL JETS

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NOMENCLATURE

C, a curvature parameter ($\equiv d/S$);
c_p, specific heat at constant pressure;
d, diameter of the rod;
h, local heat-transfer coefficient;
K, a mixing-length constant;
l, the mixing length;
p, pressure;
q, heat flux;
r, distance from the axis of symmetry;
r_I, radius of inner boundary of the wall jet (i.e. the radius of the rod);
Re_S, a Reynolds number ($\equiv \rho u_S S/\mu$);
S, slot height;

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